Superposition of perpendicular harmonic oscillations

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Superposition of perpendicular harmonic oscillations

4.1 Introduction

Our discussion has so far been confined to the superposition of two harmonic oscillations in one dimension.Let us consider the situation where two harmonic oscillations are perpendicular to each other. The familiar example is the motion of a simple pendulum whose bob can swing in the plane. We displace the pendulum in the direction and as we release it, we give it an impulse in the y- direction. How does such a pendulum oscillate? Clearly this would be a composite motion whose maximum - displacement occurs when y-displacement is zero and y-velocity is maximum and similarly maximum y- displacement occurs when -displacement is zero and -velocity is maximum. We call such an arrangement a spherical pendulum. The frequency of the superposed simple harmonic motions will obviously be the same as it depends only on acceleration due to gravity and the length of the cord. Let us study in detail the general trajectory followed by such a composite motion.

Animation on Spherical Pendulum

To view the composite motion of a spherical pendulum, the reader is recommended to visit the following web site:

Link: ***www.youtube.com/watch?v=7oRm4CtYn0Q***

Animation: Projections of Spherical pendulum

Link: http://www.youtube.com/watch?v=L4_mFNUdBrg&NR=1

Animation: Projections of Spherical pendulum

Link: http://www.youtube.com/watch?v=6hCLkTENfSA&NR=1

4.2 Superposition of two perpendicular harmonic oscillations having equal frequencies

Consider two mutually perpendicular oscillation having the same frequency and amplitudes a_1 and a_2 such that $a_1 > a_2$. These are represented by the equations,

$$x = a_1 \cos \omega_o t \quad (4.1)$$

$$y = a_2 \cos (\omega_o t + \phi) \quad , \quad (4.2)$$

where the initial phase along -axis is zero and along y-axis is Ø. Let us find out the resultant oscillation for some specific values of phase difference Ø.

(A) Analytical Method

Case I.

When $\phi = 0$ or π.

For $\phi = 0$,

$$x = a_1 \cos \omega_o t$$
$$y = a_2 \cos \omega_o t$$

i.e

$$y = \frac{a_2}{a_1} x \tag{4.3}$$

Similarly, for $\phi = \pi$,

$$x = a_1 \cos \omega_o t$$
$$y = -a_2 \cos \omega_o t$$

i.e

$$y = -\frac{a_2}{a_1} x \tag{4.4}$$

Thus we find from eqns. (4.3) and (4.4) that the resultant motion of the particle is along a straight line passing through the origin. [Recall, the equation of a straight line is y = m x + c, where in the present case, c = o and the slope, m, is positive in the first case and negative in the second case]

ForØ =0 , the motion is along the diagonal OB and for Ø = TT , the motion is along OE as is shown in the figure (4.1)

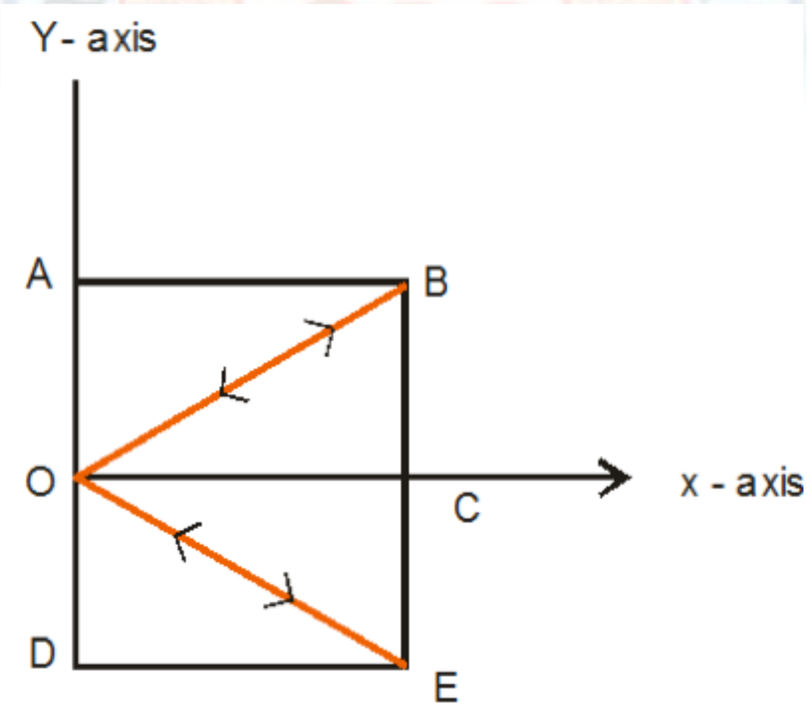


Fig.4.1

Case II.

When $\phi = \pi/2$

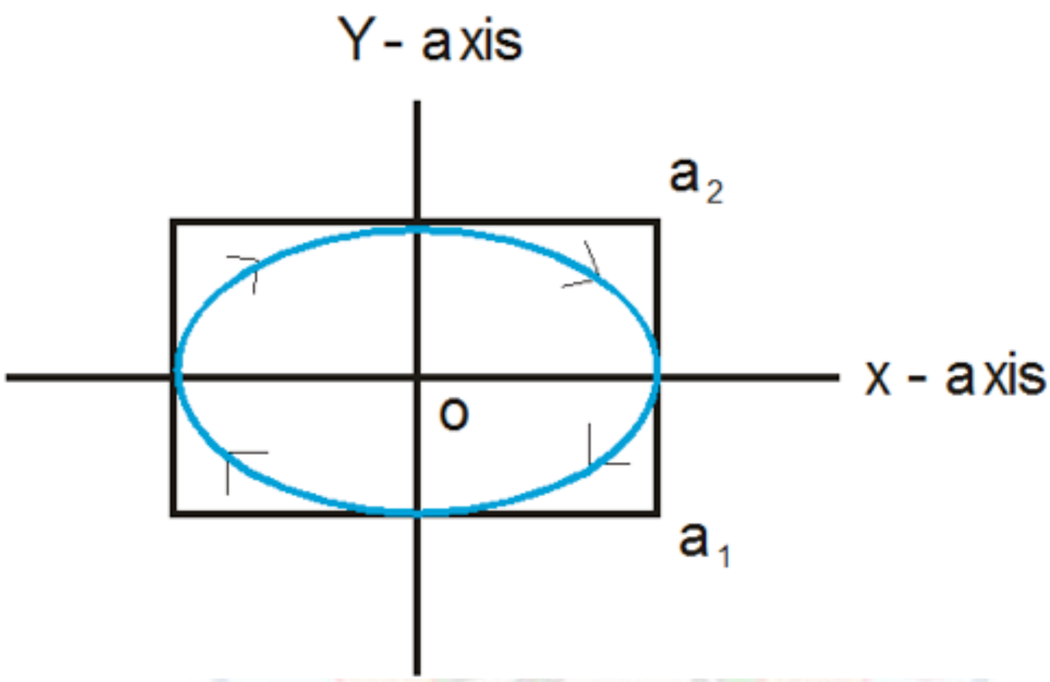
In this case the two vibrations are given by

$$x = a_1 \cos \omega_o t \tag{4.5}$$

and $$y = a_2 \cos(\omega_o t + \pi/2) = - a_2 \sin \omega_o t \tag{4.6}$$

On squaring and adding the two expressions, we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = \cos^2 \omega_o t + \sin^2 \omega_0 t = 1 \tag{4.7}$$



This is the standard equation of an ellipse-- showing that the resultant motion of the particle is along an ellipse whose principal axes lie along - and y-axis. The semi major and semi-minor axes of the ellipse are a_1 and a_2. From eqns. (4.6) and (4.7) we can see that as time increases, decreases from its maximum value a_1 but y becomes more and more negative. Thus the ellipse is described in the clockwise direction as is shown in the figure (Fig. 4.2). However, if we analyze the case forØ = 3TT/2 or Ø = -TT/2 , we shall get the same ellipse but the motion will now be in anticlockwise direction (See Fig. 4.3).

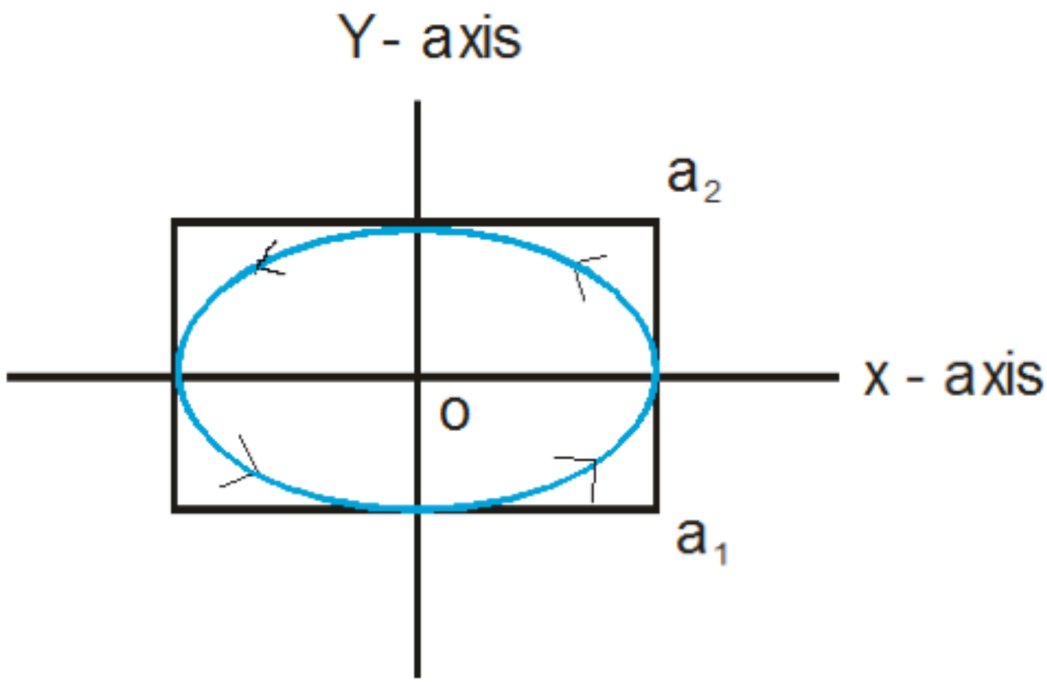


Fig.4.3

When the two amplitudes a_1 and a_2 are equal, i.e. $a_1 = a_2 = a$, equation (4.7) reduces to

$$x^2 + y^2 = a^2 \tag{4.8}$$

This equation represents a circle of radius a. Thus the ellipse reduces to a circle in this case.

Let us now come back to the general case. Considering the Eqs.(4.1) and (4.2), we write eqn. (4.2) as

$$y/a_2 = \cos(\omega_o t + \phi) = \cos\omega_o t \cos\phi - \sin\omega_o t \sin\phi \tag{4.9}$$

From eqn. (4.1), $\cos\omega_o t = x/a$, so that $\sin\omega_o t = \sqrt{1 - x^2/a_1^2}$

Substituting for $\cos\omega_o t$ and $\sin\omega_o t$ in Eq. (4.9), we get

$$y/a_2 = \frac{x\cos\phi}{a_1} - \sqrt{1 - x^2/a_1^2}\ \sin\phi$$

or

$$x/a_1 \cos\phi - y/a_2 = \sqrt{1 - x^2/a_1^2}\ \sin\phi$$

On squaring both the sides of this equation, we find

$$x^2/a_1^2 + y^2/a_2^2 - 2\frac{x\,y}{a_1\,a_2}\cos\phi = \sin^2\phi \tag{4.10}$$

This is an equation describing an ellipse whose axes are inclined to the coordinate axes. It would be instructive to show the resultant trajectories for some typical values of . These trajectories can be easily demonstrated on a cathode ray oscilloscope (CRO). A few trajectories are shown below:

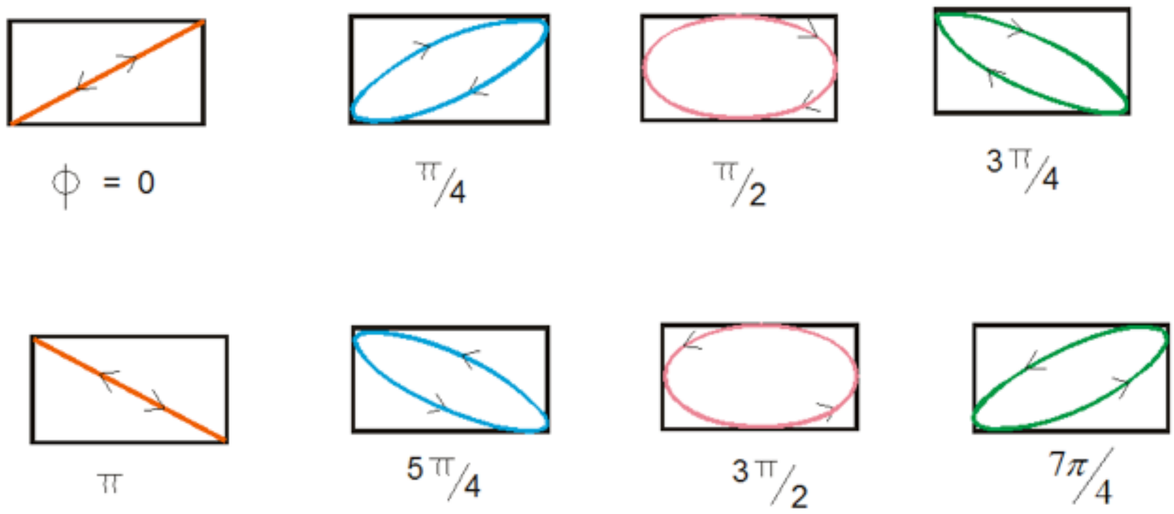


Fig.4.4

(B) Graphical Method:

We can also use graphical method employing a double application of rotating vector technique to obtain the above results. Consider two perpendicular SHMs given by

$$x = a_1 \cos \omega_o t \tag{4.11}$$

$$y = a_2 \cos (\omega_o t + \phi) . \tag{4.12}$$

Draw two circles of radii, a_1 and a_2- the circle of radius $a_1$defining the SHM along the x-axis and that of radius a_2 along the y-axis, as is shown in Fig.(4.5).

Suppose O_1P_1 represents the position of the rotating vector at a certain instant t and its projection (OX) on the x-axis gives the instantaneous displacement x, given by Eq.(4.11).

Superposition of perpendicular harmonic oscillations

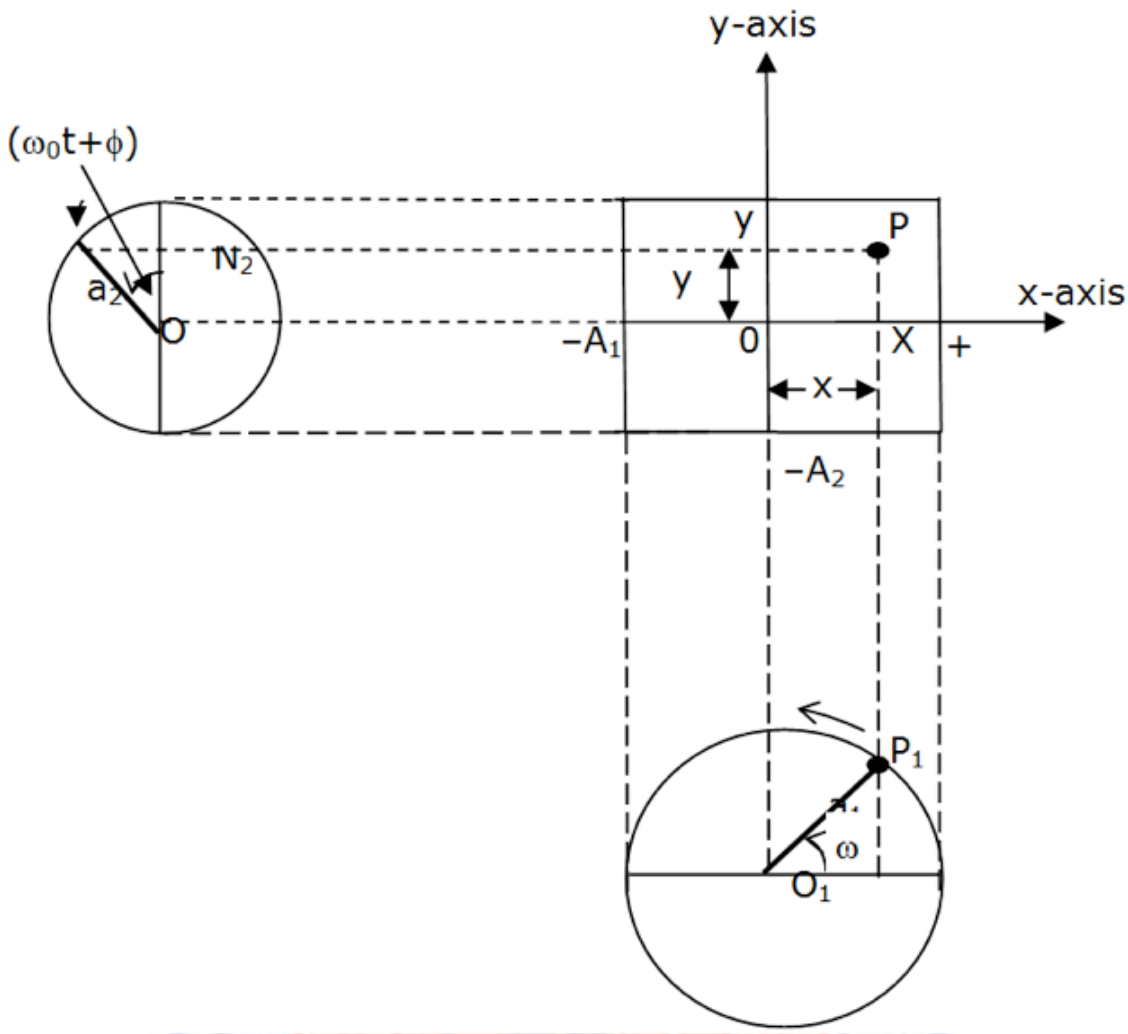


Fig. 4.5 Geometrical representation of the Superposition of two SHMs at right angles to each other.

Similarly, the projection of the rotating vector O_2P_2on the y-axis (OY) gives the instantaneous perpendicular displacement y, given by Eq.(4.12). Note that at any instant of time t, the displacements x=OX

and y=OY represent two perpendicular SHM's. When a particle is subjected to both the SHMs simultaneously, the resultant displacement at time t would be OP.

The point P is the intersection of the two perpendiculars drawn from P_1 and P_2 through the x- and y axes respectively.

The trajectory followed by the point P as a function of time t describes the resultant motion. Let us now consider a few specific cases to construct the resultant motion by choosing some values of the phase difference Ø.

Case (a); Phase Difference Ø =0:

In this case, the two perpendicular motions are

$$x = a_1 \cos \omega_o t$$
$$y = a_2 \cos \omega_o t$$

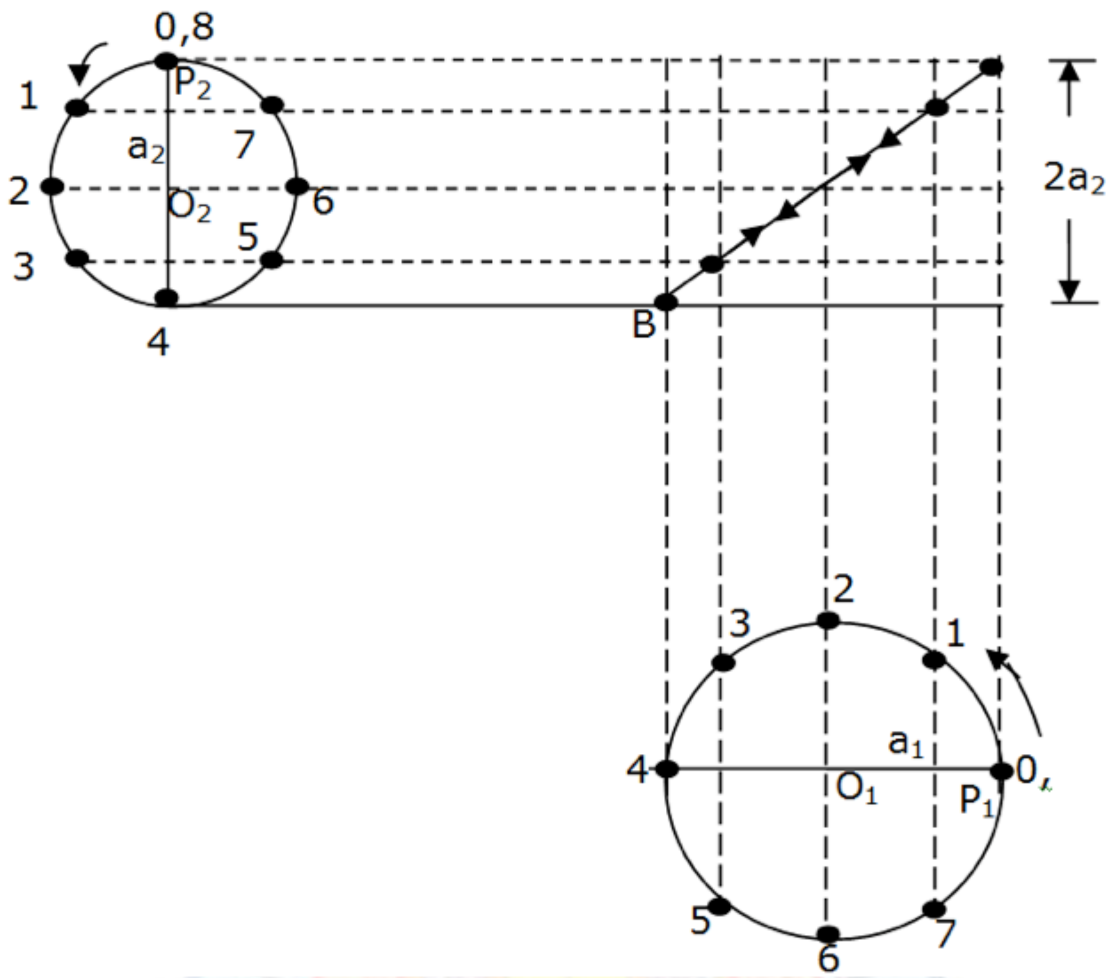


Fig. 4.6 Superposition of two perpendicular SHMs of the same frequency and zero phase difference

Let us divide the circumference of each reference circle into the number of equal parts, say, eight, as is shown in Fig.(4.6). Since the frequency of the two SHMs is the same, the rotating vector in each reference circle will describe each part in the same time, i.e.

$\pi / 4\omega_0 (= T/8).$,

The points are numbered 0, 1, 2, … beginning with t=0 when O_1P_1 is parallel to the x-axis and O_2P_2parallel to the y-axis , as the phase difference Ø between two SHMs is zero. The projections from the corresponding positions of P_1 and P_2 then result in a set of intersections representing the instantaneous positions of the point P within the rectangle of $2a_1$ and $2a_2$

sides 2. The locus of these points describes a straight line AOB with a positive slope.

Case (ii); Phase Ø = $\pi / 4$

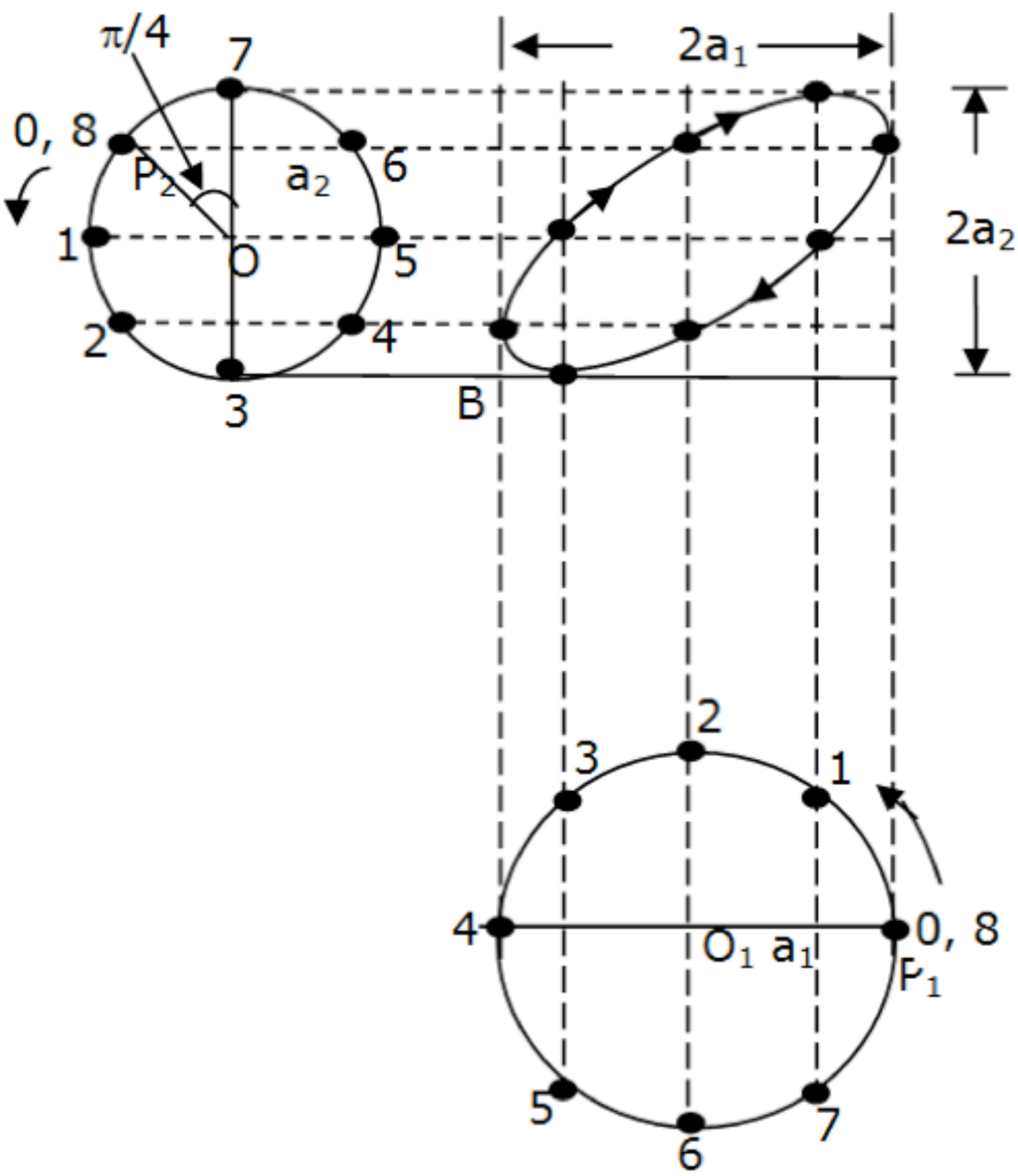


Fig. 4.7 Superposition of two perpendicular SHMs of the same frequency and phase difference of p/4._

In this case, the points 0, 1, 2, … are numbered starting with t=0 when O_1P_1 is parallel to the x-axis while the vector O_2P_2 is at angle Ø = $\pi/4$ from the y=axis, measured in counterclockwise sense as shown in Fig.(4.7). The projections of these corresponding positions of P_1 and P_2 give us a set of points of intersection, as shown in Fig.(4.7). These points of intersections represent the instantaneous positions of the point P as it moves within the rectangle of sides 2. The locus of these points is an ellipse, which is described in the clockwise sense, and is makes an angle of $\pi/4$ with the x-axis. Note that the final shape obtained can be improved if instead of dividing the circumference of the reference circles we had divided these in 16 or 32 equal parts.

Value Addition: Activity
Heading: Graphical Activity

Body:

In a cathode ray oscilloscope, the deflection of electrons by the superposition of two mutually perpendicular electric fields given by

$$x = 3\cos(\omega_o t)$$

and
$$y = 3\cos(\omega_o t + \pi/3)$$

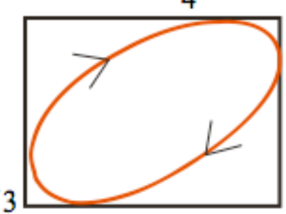
Draw the resultant trajectory of the electrons. Use also the graphical method to draw the resultant trajectory.

Solution: Using Eq.(4.10)

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{2xy}{a^2}\cos\pi/3 = \sin^2\pi/3$$

$$\frac{x^2}{9} + \frac{y^2}{9} - \frac{xy}{9} = \frac{3}{4}$$

$$x^2 + y^2 - xy = \frac{27}{4}$$



$\phi = \pi/3$

This is again an equation of ellipse whose axes are inclined to the coordinate axes by an angle of $\pi/3$. Follow the procedure described above to draw he trajectory using graphical method.

4.3 Superposition of Two (Rectangular) Mutually Perpendicular Harmonic Oscillations of Different Frequencies

Lissajous Figures

Let us now study the case when the two orthonormal harmonic oscillations have different frequencies. Here the resulting motion is more complex. This is because the relative phase, $\phi = \omega_2 t - \omega_1 t + \phi_o$ between the two vibrations will now be time dependent and would therefore gradually change with time. This makes the shape of the figure to undergo a slow change. The patterns which are thus traced out are called Lissajous figures.

Lissajous figures can be traced on a cathode ray oscilloscope when two different alternating sinusoidal voltages are applied on the defection plates, XX and YY, of the CRO. The electron beam would then trace the resultant trajectory on the fluorescent screen.

Frequencies in the ratio 1:2

(A) Analytical Method:

Let us study the situation when two perpendicular harmonic vibrations represented by

$$x = a_1 \cos(2\,\omega_o t + \phi_1) \qquad (4.13)$$

and

$$y = a_2 \cos\ \omega_o t\ , \qquad (4.14)$$

where the two frequencies are in the ratio 2:1. To find the resultant motion, we consider three cases:

Case (i); when $\phi_1 = 0:$

$$x = a_1 \cos 2\,\omega_o t = a_1\,(2\cos^2\,\omega_o t - 1) \quad and \quad y = a_2 \cos \omega_o t$$

This gives $\dfrac{x}{a_1} = \dfrac{2y^2}{a_2^{\,2}} - 1.$

Let us arrange this equation as

$$y^2 = \frac{a_2^{\,2}}{2a_1}\,(x + a_1) \qquad (4.14)$$

This is an equation of a parabola , which can be traced as is shown in the figure Fig.(4.8).

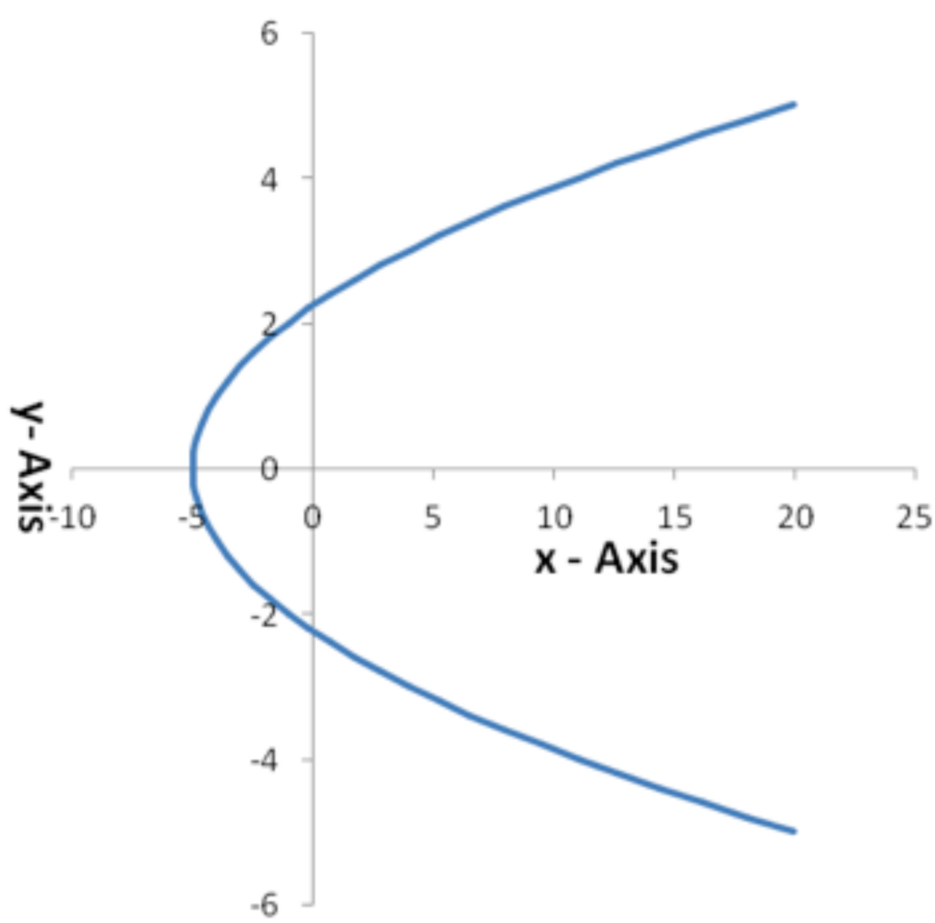


Fig.(4.8)

Case (ii) when Ø = TT/2

$$x = -a_1 \sin 2\omega_o t \quad , \text{ or } \quad -\frac{x}{a_1} = 2\sin\omega_o t \cos\omega_o t$$

and $$y = a_2 \cos\omega_0 t \quad or \quad \cos\omega_o t = {y}/{a_2} \quad and \quad \sin\omega_o t = \sqrt{1 - {y^2}/{a_2^{\,2}}}$$

Substituting for $\cos\omega_o t$ and $\sin\omega_o t$ in the equation for x, we get

$$-\frac{x}{a_1} = 2\frac{y}{a_2} \sqrt{1 - {y^2}/{a_2^{\,2}}}$$

On squaring both the sides, we have

$$4\frac{y^2}{a_2^2}\left(\frac{y^2}{a_2^{\,2}} - 1\right) + \frac{x^2}{a_1^{\,2}} = 0$$

, or

This is an equation which is fourth order in y and two order in .

Therefore, it is expected to have, in general, four roots in y (intersecting parallel to y-axis at 4 points) and two roots in (intersecting two points on lines parallel to -axis). Thus, this equation represents a figure of '8' (Fig. (4.9)).

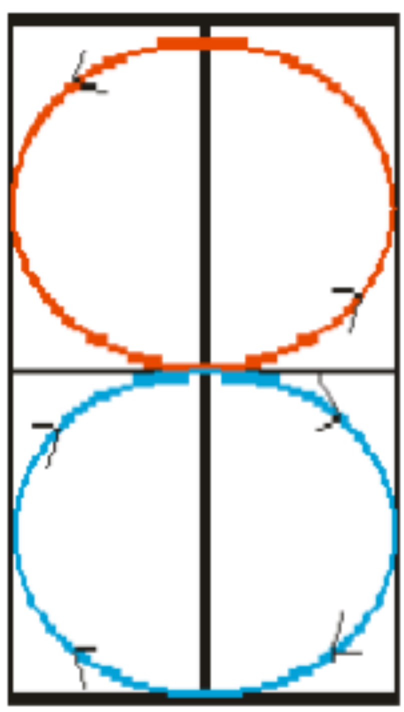


Fig.4.9

Case (iii) when $\phi_1 = \pi$:

$$x = -a_1 \cos 2\,\omega_o t \quad \text{and} \quad y = a_2 \cos \omega_o t$$

i.e.

$$-\frac{x}{a_1} = 2\cos^2 \omega_o t \; -1$$

$$= 2\, y^2 / a_2^2 \; -1$$

or

$$y^2 = -\frac{a_2^2}{2a_1}(x - a_1)$$

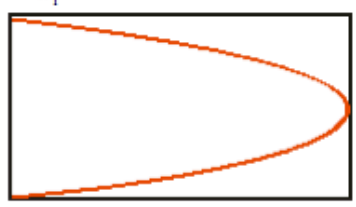


Fig.4.10

This equation again represents a parabola (Fig.4.10) but just opposite to the case shown in Fig.(4.8) when $\phi_1 = 0$.

(B). Graphical Method:

The analysis given above shows clearly that the analytical method becomes much involved for values of phase constant other than zero. Using the graphical method we shall find, that the resultant motion can be constructed quite conveniently. Let us consider the case when the two SHMS are given by

$$x = a_1 \cos \omega_o t$$
$$y = a_2 \cos (2\omega_o t + \phi_2)$$

Fig.(4.11) given below shows how the rotating vector technique is used to obtain the shape of the Lissajous figure when $\phi_2 = \pi/4$ and the two frequencies are in the ratio 1:2. The rotating vector O_2P_2 makes an angle $\pi/4$ at time t=0 with the y-axis to show that y oscillation has an initial phase of $\pi/4$. However, at this instant of time the rotating vector O_1P_1 just coincides with the x-axis to represent that the x oscillation has initial phase zero. Since the y oscillation has frequency twice that of the x-oscillation, we, therefore, choose to divide the circumference of the circle of radius into 8 equal parts and that of circle of radius into 16 equal parts. This is to ensure that during the time the vector O_1P_1 describes one-eighth part of the circle, the vector describes during this time only 1/16th of its circle. Note that while the frequency of rotation of vector is double that of rotation vector of O_2P_2, its period is just half that of . During one complete cycle of ω_2 we go through only half a cycle of ω_1 and therefore the points on the reference circles are marked accordingly. Indeed one must go through a complete cycle of ω_2 in order to get one complete period of the combined motion.

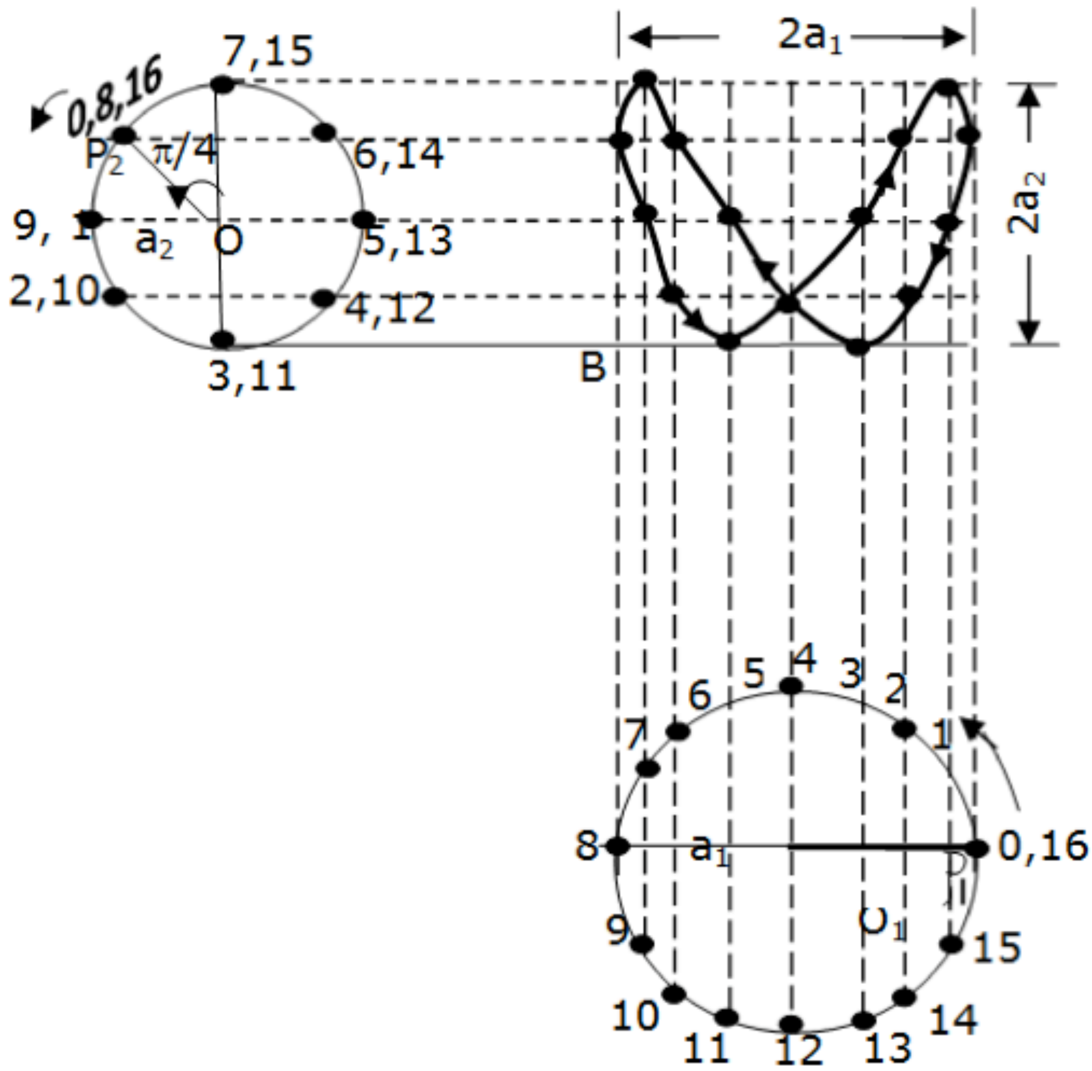


Fig. 4.11 Superposition of two perpendicular SHMs with the frequency in the ratio 1:2 and phase difference equal to $\pi / 4$.

It is now a simple exercise to construct the resultant motion corresponding to other phase differences. The following figure (Fig.(4.12)) shows the sequence of these motions for values of phase $\emptyset_2$ in the range 0 to π.

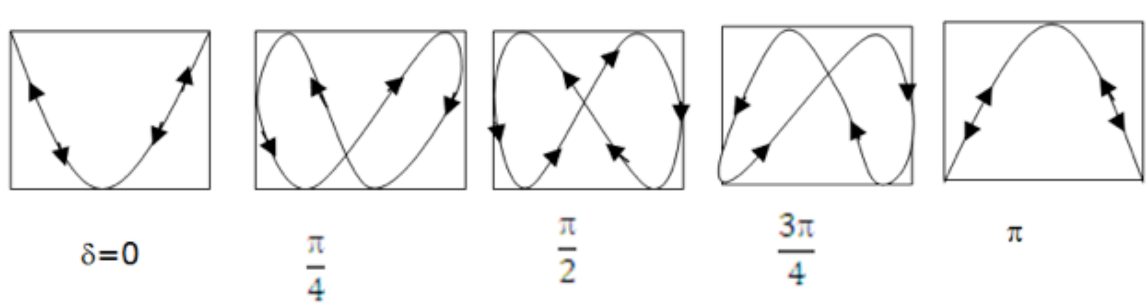


Fig. 4.12 Lissajous figures: $w_2=2w_1$ with various initial phase differences

Superposition of perpendicular harmonic oscillations

Value Addition: Biographical Sketch
Heading: Biography of Jules Antoine Lissajous
Body:

Jules Antoine Lissajous

Lissajous was a French physicist and mathematician who was born on March 4,1822 at Versaillas, France. Lissajous graduated from the Ecole Normale Superieure in 1847 in Paris. He is known for many innovations but perhaps the best known is a special apparatus – the Lissajous apparatus--- that creates the variety of figures which are now named after him as the Lissajous figures. This apparatus made him realize, in a way, his idea of making ‘sound waves visible’!. This apparatus uses two tuning forks with mirrors attached to them , mounted in mutually perpendicular directions. By making the light reflected from the mirror of the first tuning fork , to fall on the mirror of the second fork and then getting the light reflected from this on the screen, one can see the resulting Lissajous figures on the screen. It is possible to vary the frequency ratio and the phase difference between the two oscillations and see the effects of these on the patterns of the Lissajous figures obtained.

Lissajous apparatus led to the invention of other apparatus such as the harmonograph. It is interesting to note that many of the designs, used by fabric and cloth designers, can be correlated with Lissajous figures corresponding to different mutual orientations and to different frequency, phase and amplitude ratios of the two superimposed simple harmonic motions.

Value Addition: Animation
Heading: Lissajous Figures Activity

Animation Idea

Show a mirror attached to one of the prongs of a tuning fork.

Let a beam of light fall on this mirror.

Let the reflected beam fall on the prong of a second tuning (identical) fork oriented perpendicularly and again having a mirror attached to it.

Superposition of perpendicular harmonic oscillations

Let the reflected beam now fall on a screen.

Make both the tuning forks vibrate simultaneously.

See the figure on the screen.

Change the figure by (i) making one fork vibrate and (ii) letting the other fork vibrate a little late (vary this interval and get different figures).

Summary

In this chapter, the reader learns to

apply the principle of superposition to two mutually perpendicular harmonic oscillations of equal frequency for different phases using analytical method;

use graphical method to study the resultant motion for different phases;

study superposition of two mutually perpendicular SHMS of frequencies in the ratio 1:2 and of different phases analytically. This enables the reader to study the formation of Lissajous figures;

apply graphical method to trace the trajectory of the resultant motion of two perpendicular SHMs of frequencies in the ratio1:2 for phase differenc zero and *TT*/4.

Problems

1. Two tuning forks A and B of frequencies close to each other are used to obtain Lissajous figure. It is observed that the figure goes through a cycle of changes in 20s. If A is loaded slightly with wax, the figure goes through a cycle of changes in 10s. If the frequency of B is 256 Hz, what is the frequency of A before and after loading ?

Solution :

Period of 20 s corresponds to the frequency of 0.05 Hz.

$$\nu_A - \nu_B = \pm 0.05\, Hz$$

On loading the prong of A with wax, the frequency of A will decrease. But the cycle of changes is completed in 10 s , i.e the frequency difference increases to 0.1Hz. Therefore the frequencies of A before and after loading are respectively 256 -0.05 = 255.95 Hz and 256 – 0.1 = 255.9 Hz.

2. When two simple harmonic motions

$$x = a_1 \cos\omega_0 t \qquad and \qquad y = a_2 \cos(\omega_0 t + \phi)$$

are superimposed, what is (a) the basic difference in the two trajectories when ; and (b) when ?

$\phi = \pi/2$ *and* $-\pi/2$; and (b) when $a_1 > a_2$ *and* $a_1 < a_2$?

Solution. (a) In both the cases the figure is an ellipse. However, when $\phi = \pi/2$, the ellipse is described in the clockwise direction and when $\phi = \pi/2$, it is in the anti-clockwise direction.

(b) When $a_1 > a_2$ it implies semi major x-axis a_1 is greater than the semi- minor y-axis a_1 and $a_1 < a_2$ for semi major axis is along the y-axis and semi minor axis is along the x-axis.

3. In a cathode ray oscilloscope, the deflection of electrons by the superposition of two mutually perpendicular fields is given by

$$x = 4\cos\omega_0 t \quad , \qquad y = 4\cos(\omega_0 t + \phi).$$

Analyze the resultant trajectories followed by the electrons when $\phi = \pi/4 \quad and \quad \phi = 5\pi/4$.

Solution: Using the equation (3.2.25) , the resultant trajectory follows the path given by

For

$$x^2/a_1^2 + y^2/a_2^2 - 2\frac{x\,y}{a_1\,a_2}\cos\phi = \sin^2\phi$$

For $a_1 = a_2 = 4 \quad and \quad \phi = \pi/4$, we get

$$x^2 + y^2 - 2xy\cos\pi/4 = 16\sin^2\phi$$

$$i.e, \quad x^2 + y^2 - 2xy\frac{1}{\sqrt{2}} = 8.$$

In the second case , when $\phi = 5\pi/4$

$$x^2 + y^2 - 2xy\cos 5\pi/4 = 16\sin^2\phi$$

$$i.e, \quad x^2 + y^2 + 2xy\frac{1}{\sqrt{2}} = 8.$$

The two trajectories follow the paths as given in Fig.4.4 for the cases $\phi = \pi/4 \quad and \quad \phi = 5\pi/4$.

4. Use the graphical method to trace the trajectories of the resultant motion of problem 3 given above.

5. Consider the superposition of two harmonic waves given by the equations

$$x = a\sin(\omega_o t)$$

$$y = a\sin(3\omega_o t + \phi_1), \qquad \text{When } \phi_1 = \pi/2$$

Draw the Lissajous figure representing the resultant motion.

Solution:

$$\frac{y}{a} = \sin(3\omega_0 t + \pi/2) = \cos(3\omega_0 t) = (3\cos\omega_0 t - 4\cos^3\omega_0 t)$$

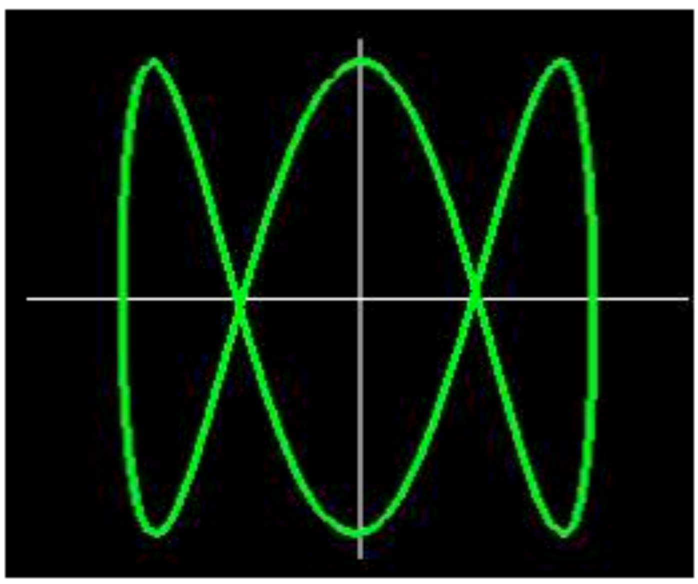
$$or \quad \frac{y}{a} = \sqrt{1-\sin^2\omega_0 t}\,(3-4(1-\sin^2\omega_0 t))$$

$$= \sqrt{1-x^2/a^2}\ (4x^2/a^2-1)$$

On squaring both sides, we get

$$\frac{y^2}{a^2} = (1-\frac{x^2}{a^2})(4\frac{x^2}{a^2}-1)^2$$

The corresponding Lissajous figure is as shown.



6. How would the figure drawn above change, if we consider the superposition of two harmonic waves given by the equations

$$x = a\cos(3\omega_0 t + \phi_1), \quad y = a\cos(\omega_0 t)\ , \quad when\ \phi_1 = \pi/2\,?$$

Solution: It is easy to show that the roles of x- and y- axes have interchanged. That means the above curve would now be rotated by 90degress.

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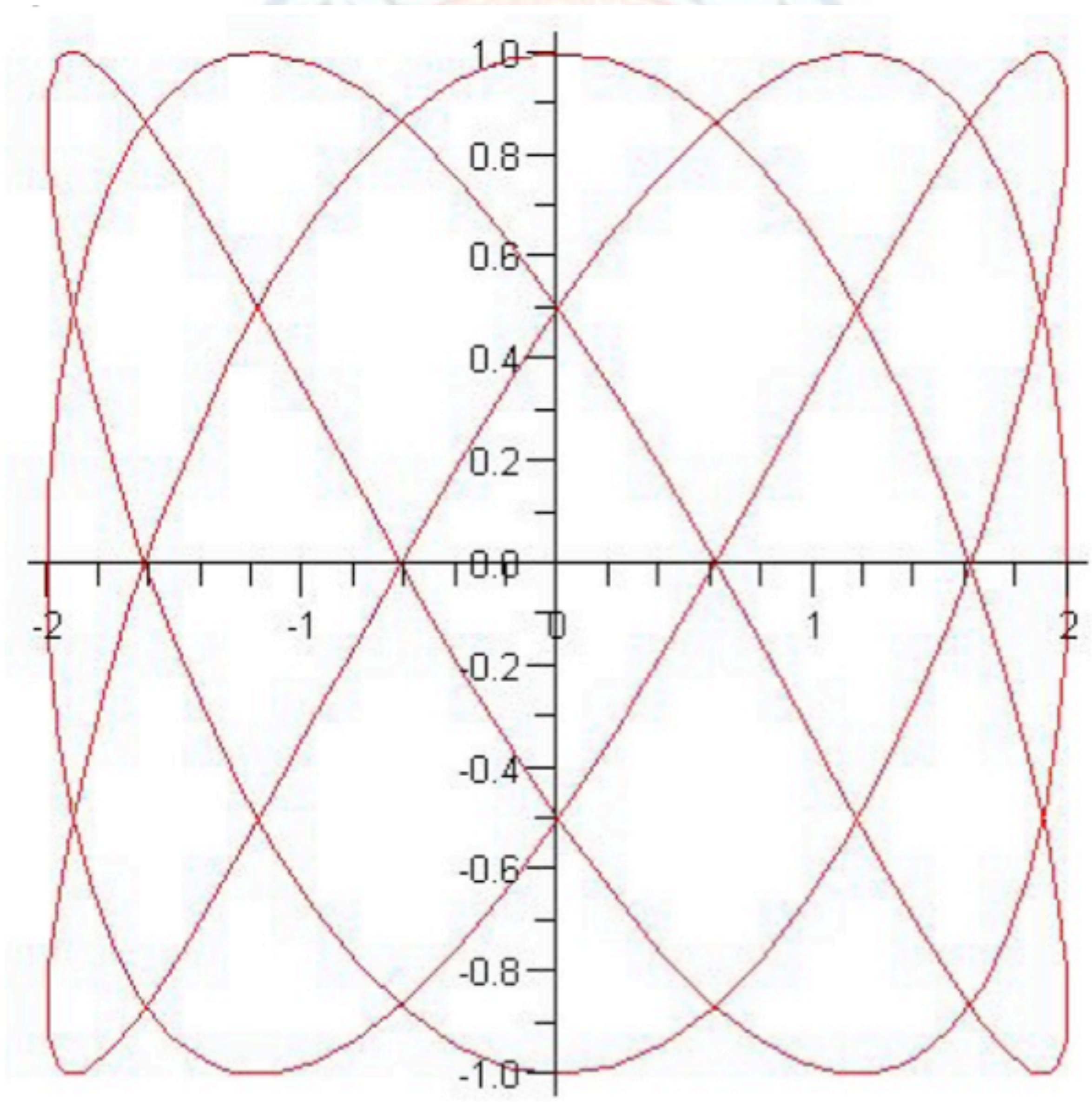
Value Addition: Animation
Heading: Lissajous figures
Body: 
Animate the figure shown above with lobes moving as shown in the link