Different types of Laws of Large Numbers:-

(1) **Bernoulli's Law of Large Numbers:-** Let there be n trials of an event, each trial resulting in a success/failures. If X is the no. of successes in n trials with constant prob^y p of success for each trial, then $E(X) = np$, $V(X) = npq$, where $q = 1-p$. The variable $\frac{X}{n}$ represents the proportion of successes and $E\left(\frac{X}{n}\right) = p$, $V\left(\frac{X}{n}\right) = \frac{pq}{n}$.

Then $P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \to 1$ as $n \to \infty \Rightarrow P\left\{\left|\frac{X}{n} - p\right| \geq \varepsilon\right\} \to 0$ as $n \to \infty$.

for any $\varepsilon > 0$. This implies that $\frac{X}{n}$ converges in prob^y to p as $n \to \infty$, i.e., $\frac{X}{n} \xrightarrow{p} p$ as $n \to \infty$.

Pf:- Applying chebychev's inequality to the r.v. $\frac{X}{n}$, we get for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X}{n} - E\left(\frac{X}{n}\right)\right| \geq \varepsilon\right\} \leq \frac{\sigma^2}{\varepsilon^2}$$

$$= \frac{pq}{n\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}$$

Here $\sigma^2 = Var\left(\frac{X}{n}\right)$
$= \frac{pq}{n}$
and $k\sigma = \varepsilon$
$\Rightarrow k = \varepsilon/\sigma$
which gives $\frac{1}{k^2} = \frac{\sigma^2}{\varepsilon^2}$

$\because$ the maximum value of pq is at $p = q = \frac{1}{2}$, i.e. $\max(pq) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, i.e. $pq \leq \frac{1}{4}$

$\because$ ε is arbitary, we have

$$P\left\{\left|\frac{X}{n} - p\right| \geq \varepsilon\right\} \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \to 1 \text{ as } n \to \infty$$

Ex :- A symmetric die is thrown 600 times. Find the lower bound for the prob^{n} of getting 80 to 120 sixes.

Sol^{n} :- Let S be the total no. of successes. Then

$$E(S) = np = 600 \times \frac{1}{6} = 100,$$

$$V(S) = npq = 600 \times \frac{1}{6} \times \left(1 - \frac{1}{6}\right) \qquad (\because q = 1-p)$$

$$= 600 \times \frac{1}{6} \times \frac{5}{6}$$

$$= \frac{250}{3}$$

Using Chebychev's inequality, we get

$$P\{|S - E(S)| < K\sigma\} \geq 1 - \frac{1}{K^2}$$

$$\Rightarrow P\left\{|S - 100| < K \cdot \sqrt{\frac{250}{3}}\right\} \geq 1 - \frac{1}{K^2}$$

$$\therefore P\left\{100 - K\sqrt{\frac{250}{3}} < S < 100 + K\sqrt{\frac{250}{3}}\right\} \geq 1 - \frac{1}{K^2}$$

We have to set $80 < S < 120$ which gives :-

$$100 - K\sqrt{\frac{250}{3}} = 80 \quad \text{or} \quad 100 + K\sqrt{\frac{250}{3}} = 120$$

$$\Rightarrow \cancel{-}K\sqrt{\frac{250}{3}} = \cancel{-}20$$

$$\Rightarrow K^2 \cdot \frac{250}{3} = 400$$

$$\Rightarrow K^2 = \frac{3 \times 400}{250} = \frac{24}{5}$$

i.e. $K = \sqrt{\frac{24}{5}}$

Thus we have, $P\{80 < S < 120\} \geq 1 - \frac{1}{\left(\frac{24}{5}\right)}$

$$= 1 - \frac{5}{24}$$

$$= \frac{19}{24} \text{ //}$$

Q) If we wish to estimate the proportion of engineers and scientists who have studied prob^y theory and you wish your estimate to be correct within 2%, with prob^y 0.95 or more, how large a sample would you take :-

① if you have no idea what the true proportion is,

② if you are confident that the true proportion is less than 0.2 ?

Soln :- We have, in Bernoulli Law of Large Numbers, we get for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X}{n} - p\right| \ngeq \varepsilon\right\} \le \frac{1}{4n\varepsilon^2}$$

$$\Rightarrow P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \geq 1 - \frac{1}{4n\varepsilon^2}$$

① In the first case, p is not known. But here $\varepsilon = 2\% = \frac{2}{100} = 0.02$

so that $P\left\{\left|\frac{X}{n} - p\right| < 0.02\right\} \geq 1 - \frac{1}{4n(0.02)^2} = 0.95$ (which is given in the question)

i.e. $0.95 = 1 - \frac{1}{4n(0.02)^2}$

$$\Rightarrow \frac{1}{4n \times 0.0004} = 0.05$$

$$\Rightarrow \frac{1}{0.0016n} = 0.05$$

i.e. $n = \frac{1}{0.05 \times 0.0016} = 12{,}500$

Thus $n = 12{,}500$ //

② Here it is given that $p < 0.2 \Rightarrow pq = p(1-p) = 0.2(1-0.2)$
$= 0.2 \times 0.8 = 0.16$

Taking $\varepsilon = 2\% = 0.02$ we have to find n so that

$$P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \geq 1 - \frac{pq}{n\varepsilon^2}$$

i.e. $P\left\{\left|\frac{X}{n}-p\right| < 0.02\right\} \geq 1-\frac{0.16}{n(0.02)^2} = 0.95$

$$\Rightarrow 0.95-1 = \frac{-0.16}{n\times 0.0004}$$

$$\Rightarrow \cancel{-}0.05 = \frac{\cancel{-}1}{n}\times 400$$

$$\Rightarrow n = \frac{400}{0.05} = 8000.$$

$\therefore$ In this case $n = 8,000$ //