Ex. Show that the set $M_2(Z)$ of 2×2 matrices with integer entries is a noncommutative ring with ~~~~ unity. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln:- $M_2(Z) = \{ A \mid A$ is a 2×2 matrix with integer entries$\}$.

Let A, B, C be any three elements of $M_2(Z)$. so all of A, B, C are 2×2 matrices with integer entries.

(i) we know matrix addition is commutative

$\therefore A+B = B+A$, for all $A, B \in M_2(Z)$

(ii) we know matrix addition is associative

$\therefore (A+B)+C = A+(B+C)$, for all $A, B, C \in M_2(Z)$

(iii) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(Z)$ and for any $A \in M_2(Z)$

$A + 0 = A$

(iv) for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$, $a, b, c, d \in Z$

$-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ and $A + (-A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

(v) We know matrix multiplication is associative

$\therefore A(BC) = (AB)C$, for all $A, B, C \in M_2(Z)$

(vi) we know matrix multiplication is distributive

$\therefore A(B+C) = AB + AC$ and $(B+C)A = BA + CA$ for all $A, B, C \in M_2(Z)$

$\therefore M_2(Z)$ is a ring with respect to matrix addition and multiplication.

Since matrix multiplication is not commutative so $M_2(Z)$ is not commutative ring.

Also, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(Z)$ and for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z)$

$a, b, c, d \in Z$.

$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unity of the ring $M_2(Z)$

Hence $M_2(Z)$ is a non commutative ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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Ex. Show that the set $2Z$ of even integers under ordinary addition and multiplication is a commutative ring without unity.

Ex. Let $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$. Prove that $Z[\sqrt{2}]$ is a ring under the ordinary addition and multiplication of real numbers.

Soln: For $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$, $a, b, c, d \in Z$.

$$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2} \in Z[\sqrt{2}]$$

Since $a + c \in Z$, $b + d \in Z$.

Also $xy = (a + b\sqrt{2})(c + d\sqrt{2})$

$= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd$

$= (ac + 2bd) + (ad + bc)\sqrt{2} \in Z[\sqrt{2}]$

Since $ac + 2bd \in Z$ and $ad + bc \in Z$.

(i) $\therefore$ addition and multiplication are binary operations on $Z[\sqrt{2}]$.

(i) For $x = a + b\sqrt{2},\ y = c + d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$

$x + y = (a + c) + (b + d)\sqrt{2}$

$= (c + a) + (d + b)\sqrt{2}$

$= (c + d) + (a + b)\sqrt{2} = y + x$

(ii) Let $x = a + b\sqrt{2},\ y = c + d\sqrt{2},\ z = e + f\sqrt{2}$

where $a, b, c, d, e, f \in \mathbb{Z}$

$(x + y) + z =$

$x + (y + z) =$

(iii) $0 = 0 + 0\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ and if $x = a + b\sqrt{2}$

$a, b, \in \mathbb{Z}$ is any element of $\mathbb{Z}[\sqrt{2}]$, then

$x + 0 = (a + b\sqrt{2}) + (0 + 0\sqrt{2}) = (a + 0) + (b + 0)\sqrt{2}$

$= a + b\sqrt{2} = x.$

(iv) for $x = a + b\sqrt{2},\quad -x = (-a) + (-b)\sqrt{2}$

and $x + (-x) = (a + b\sqrt{2}) + \{(-a) + (-b)\sqrt{2}\}$

$= a + (-a) + \{b + (-b)\}\sqrt{2}$

$= 0 + 0\sqrt{2} = 0.$

(v) for $x = a + b\sqrt{2},\ y = c + d\sqrt{2},\ z = e + f\sqrt{2}$

$x(yz) =$

$(xy)z =$

(vi)

$x(y + z) =$

$(y + z)x =$