Ring

Definition: A ring R is a set with two binary operations, addition and multiplication such that for all a, b, c in R:

1. $a + b = b + a$
2. $(a+b)+c = a+(b+c)$
3. There is an element 0 in R such that $a + 0 = a$
4. There is an element $-a$ in R such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$
6. $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.

Note: The term 'ring' was coined in 1897 by the German mathematician David Hilbert.

Note. From the first four conditions of the definition of ring it is clear that a ring is an Abelian group under addition.

Note. In a ring multiplication need not be commutative. If in a ring multiplication is commutative then the ring is called commutative ring.

Note: A ring need not have an identity under multiplication.

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* One trivial example of ring.

Consider the set $R = \{0\}$

The set Contains only one element which is 0. All the six Conditions of definition of ring are satisfied by all elements of the set $R = \{0\}$. So it is a ring.

Note If a ring other than $\{0\}$ has an identity under multiplication, then the ring has a unity or identity.

Note. A nonzero element of a Commutative ring with unity need not have a multiplicative inverse. If a non zero element of a Commutative ring with unity has a multiplicative inverse then we say that it is a unit of the ring.

Examples of rings.

Example 1. The set Z of integers under ordinary addition and multiplication is a commutative ring with unity. The units of Z are 1 and -1.

Soln. for $a, b, c \in Z$

(1) $a + b = b + a$

(2) $(a+b) + c = a + (b+c)$

(3) $0 \in Z$ and $a + 0 = a$.

(4) for $a \in Z$, $-a \in Z$ and $a + (-a) = 0$.

5. $a(bc) = (ab)c$

6. $a(b+c) = ab + ac$

and $(b+c)a = ba + ca$

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Hence Z is a ring under ordinary addition and multiplication.

Also for $a, b \in Z$

$$ab = ba$$

$\therefore$ Z is a commutative ring.

Next, the number $1 \in Z$ and for $a \in Z$

$$a \cdot 1 = a$$

$\Rightarrow$ 1 is identity or unity element of Z.

Thus Z is a commutative ring with unity

Also, $(1)(1) = 1$ and $(-1)(-1) = 1$.

$\therefore$ Multiplicative inverse of 1 is 1 and multiplicative inverse of -1 is -1.

For $a \neq 0, 1$, the ~~inverse of a~~ multiplicative inverse of a is $\frac{1}{a}$. but $\frac{1}{a} \notin Z$ for $a \neq 1$.

Thus the units of Z are 1 and -1.

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