Theorem: If $a \in \mathbb{R}$ is such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$

Pf. If possible let $a > 0$. Take $\varepsilon_0 = \frac{1}{2}a$. Then
~~$0 < \varepsilon_0 < a$~~ $0 < \varepsilon_0 = \frac{1}{2}a < a \Rightarrow 0 < \varepsilon_0 < a$.
But this is a contradiction since $a < \varepsilon$ for every $\varepsilon > 0$. Hence $a = 0$.

Remark. If $a \in \mathbb{R}$ is such that $0 \le a \le \varepsilon$ for every $\varepsilon > 0$ then $a = 0$.

Theorem If $ab > 0$, then either
(i) $a > 0$ and $b > 0$ or
(ii) $a < 0$ and $b < 0$.

Pf:- Let $ab > 0$. Then $a \ne 0$ and $b \ne 0$, because if any one of a and b is zero then ab will be zero [$\because$ ~~$x.0=$~~ $x.0 =$ for all $x \in \mathbb{R}$)
Since $a \ne 0$ so by Trichotomy property either $a > 0$ or $a < 0$. If $a > 0$, then $\frac{1}{a} > 0$.

$\therefore \left(\frac{1}{a}\right) \cdot ab > 0$ [by 2nd order property]

$\Rightarrow b > 0$.

Similarly if $a < 0$, then $\frac{1}{a} < 0$ ~~i.e. $-a \in P$, $-\frac{1}{a} \in P$~~

$\therefore$ ~~$\left(-\frac{1}{a}\right)(-$~~ $\left(\frac{1}{a}\right)(ab) < 0$

$\Rightarrow b < 0$.

Absolute Value of real numbers

Definition: The absolute value of a real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}$$

eg. $|5| = 5$, $|-7| = 7$.

Note. From the def^n it is clear that $|a| \geq 0$ for all $a \in \mathbb{R}$. and $|a| = 0$ if and only if $a = 0$.

Note. $|-a| = |a|$ for all $a \in \mathbb{R}$.

If $a = 0$ then $|a| = |0| = 0$ and $|-a| = |-0| = |0| = 0$

$\therefore |a| = |-a|$

If $a > 0$, then $-a < 0$.

$\therefore |a| = a$ and $|-a| = -(-a) = a$

$\therefore |a| = |-a|$

If $a < 0$ then $-a > 0$

$\therefore |a| = -a$ and $|-a| = -a$.

$\therefore |-a| = |a|$

Theorem: (a) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

(b) $|a|^2 = a^2$ for all $a \in \mathbb{R}$

(c) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

Pf:- (a) **Case I.** If $a = 0$ then $|ab| = |0.b| = |0| = 0$ and $|a||b| = |0||b| = 0.|b| = 0$.

$\therefore |ab| = |a||b|$

Similarly **Case 2.** If $b = 0$ then $|ab| = |a||b|$

Case 3. If $a > 0$, $b > 0$ then $ab > 0$

$\therefore |ab| = ab = |a||b|$

Case 4. If $a > 0$, $b < 0$ then $ab < 0$

$\therefore |ab| = -ab = a.(-b) = |a||b|$

Case 5

~~Similarly~~ If $b > 0$, $a < 0$ then $a.b < 0$

$\therefore |ab| = -ab = (-a)b = |a||b|$

Case 6. If $a < 0$, $b < 0$ then $ab > 0$

$\therefore |ab| = ab = (-a)(-b) = |a||b|$

(b) Since $a^2 \geq 0$ for all $a \in \mathbb{R}$

$\therefore a^2 = |a^2| = |a.a| = |a||a| = |a|^2$

(c) If $|a| \leq c$ then $a \leq c$ for $a \geq 0$

and $-a \leq c$ for $a < 0$.

$\Rightarrow a \geq -c$

$\therefore -c \leq a \leq c$.

Conversely if $-c \leq a \leq c$ then $-c \leq a$ and $a \leq c$

i.e. $a \leq c$ and $-a \leq c$.

$\Rightarrow |a| \leq c$.

(d). Since $|a| \geq 0$ for all $a \in \mathbb{R}$ so taking

$c = |a|$ in part (c) we get

$-|a| \leq a \leq |a|$