Home work.

1. If $a, b \in \mathbb{R}$, prove the following

(a) If $a + b = 0$ then $b = -a$

(b) $-(-a) = a$ (c) $(-1)a = -a$

(d) $(-1)(-1) = 1$.

The order properties of $\mathbb{R}$.

There is a nonempty subset P of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:

(i) If a, b belong to P then $a+b$ belongs to P

(ii) If a, b belong to P then ab belongs to P.

(iii) If a belongs to $\mathbb{R}$, then exactly one of the following holds:

$$a \in P, \quad a = 0, \quad -a \in P.$$

Note. property (iii) is called Trichotomy property.

Note. If $a \in P$, we write $a > 0$ and say that a is a positive real number.

If $a \in P \cup \{0\}$, we write $a \geq 0$ and say that a is a nonnegative real number.

If $-a \in P$, we write $a < 0$ and say that a is a negative real number.

If $-a \in P \cup \{0\}$, we write $a \leq 0$ and says that a is nonpositive real number.

Defn. Let a, b be elements of $\mathbb{R}$.

(a) If $a - b \in P$ then we write $a > b$ or $b < a$

(b) If $a - b \in P \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

Note Trichotomy property implies that for ~~a, b, c~~, $a, b \in \mathbb{R}$, exactly one of the following will hold;
$a > b$, $a = b$, $a < b$

Theorem: Let a, b, c be any element of $\mathbb{R}$.

(i) If $a > b$ and $b > c$ then $a > c$.

(ii) If $a > b$, then $a + c > b + c$

(iii) If $a > b$ and $c > 0$ then $ca > cb$
If $a > b$ and $c < 0$ then $ca < cb$.

Pf. (i) suppose $a > b$ and $b > c$
Then $a - b \in P$ and $b - c \in P$.
By first order property we get
$(a-b) + (b-c) \in P$
$\Rightarrow a - c \in P \Rightarrow a > c$.

(ii) Suppose $a > b$. Then $a - b \in P$.
Now, for $c \in \mathbb{R}$
$(a+c) - (b+c) = a - b \in P$
$\Rightarrow (a+c) - (b+c) \in P$
$\Rightarrow a + c > b + c$

(iii) Suppose $a > b$ and $c > 0$. Then $a - b \in P$ and $c \in P$.
$\therefore$ by 2nd order property, $(a-b)c \in P \Rightarrow ac - bc \in P$
$\Rightarrow ac > bc \Rightarrow ca > cb$.
Next, suppose $a > b$ and $c < 0$. Then $a - b \in P$ and $-c \in P$
$\therefore -c(a-b) \in P \Rightarrow cb - ca \in P$
$\Rightarrow cb > ca \Rightarrow ca < cb$

Theorem

~~Ex~~: (a) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$

(b) $1 > 0$, (c) If $n \in \mathbb{N}$ then $n > 0$.

Pf. (a) By Trichotomy property if $a \neq 0$ then either $a > 0$ or $a < 0$ i.e. either $a \in P$ or $-a \in P$.

If $a \in P$ then $a \cdot a = a^2 \in P$ [by 2nd order prop. of $\mathbb{R}$]

If $-a \in P$ then $(-a) \cdot (-a) = a^2 \in P$.

Thus if $a \neq 0$ so $a^2 > 0$.

(b) Since $1 = 1^2$ so by using result of part (a), $1 > 0$.

(c) ~~If~~ we need to prove, if $n \in \mathbb{N}$ then $n > 0$.

we will Mathematical induction to prove it.

For $n = 1$, $\because 1 > 0$ so the result is true for $n = 1$. Let the result be true for $n = k$.

i.e. $k > 0$.

Now for $n = k + 1$, $k + 1 > 0$ because

$k > 0$, $1 > 0 \Rightarrow k \in P$, $1 \in P \Rightarrow k + 1 \in P$

$\Rightarrow k + 1 > 0$

Thus the result is true for $n = k + 1$.

Hence by Mathematical induction the result is true for all n i.e. $n > 0$ if $n \in \mathbb{N}$.