Algebraic properties of $\mathbb{R}$

On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication respectively. These operations satisfy the following properties:

A1) $a+b = b+a$, $\forall a, b \in \mathbb{R}$ [Commutative property of addition]

A2) $(a+b)+c = a+(b+c)$, $\forall a, b, c \in \mathbb{R}$ [Associative prop. of addition]

A3) There exists an element 0 in $\mathbb{R}$ such that

$0+a = a$ and $a+0 = a$ for all $a \in \mathbb{R}$

[existence of zero element]

A4) For each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a+(-a) = 0$ and $(-a)+a = 0$.

[existence of negative element]

M1) $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}$ [Commutative property of multiplication]

M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a, b, c \in \mathbb{R}$ [Associative prop. of multiplication]

M3) There exists an element 1 in $\mathbb{R}$ distinct from 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$. [existence of a unit element

M4) For each $a \neq 0$ in $\mathbb{R}$, there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot \left(\frac{1}{a}\right) = 1$ and $\left(\frac{1}{a}\right) \cdot a = 1$

[existence of inverse element]

D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$, $\forall a, b, c \in \mathbb{R}$

[Distributive property of multiplication over addition]

Theorem: (a) If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$

(b) If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$ then $u = 1$

(c) If $a \in \mathbb{R}$ then $a \cdot 0 = 0$

Pf: (a) Let $z + a = a \longrightarrow (1)$

Now, $z = z + 0$ [by A3]

$= z + (a + (-a))$ [by A4]

$= (z + a) + (-a)$ [by A2]

$= a + (-a)$ [from (1)]

$= 0$ [by A4]

(b) Let $u \cdot b = b \rightarrow (11)$, $b \neq 0$

Now, $u = u \cdot 1$ [by M3]

$= u \cdot (b \cdot (\frac{1}{b}))$ [by M4]

$= (u \cdot b) \cdot (\frac{1}{b})$ [by M2]

$= b \cdot (\frac{1}{b})$ by (11)

$= 1$ [by M4]

(c)

$a + a \cdot 0 = a \cdot 1 + (a \cdot 0)$ [by M3]

$= a \cdot (1 + 0)$ [by D]

$= a \cdot 1$ [by A3]

$= a$ [by M3]

$\Rightarrow a + a \cdot 0 = a$

$\Rightarrow a \cdot 0 = 0$ [by part (a) of this theorem]