Wave and Optics

Unit I: Superposition of Collinear Harmonic Oscillations

By

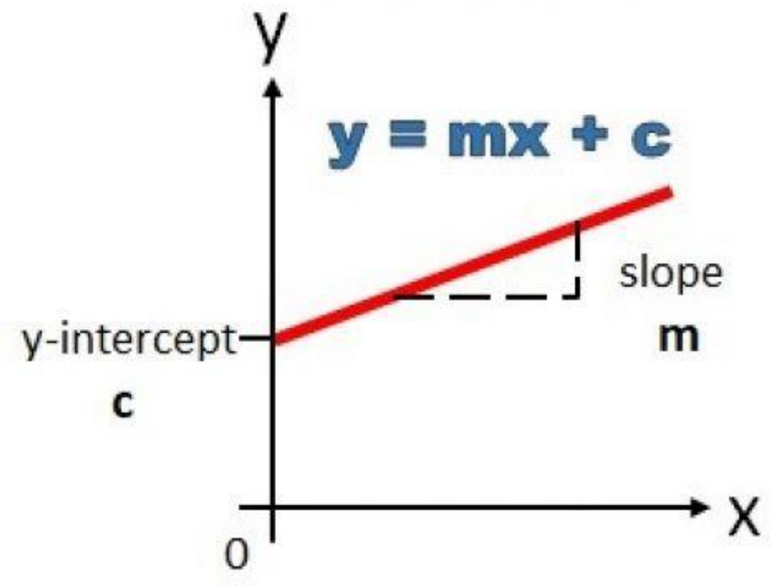
Namita Ojah

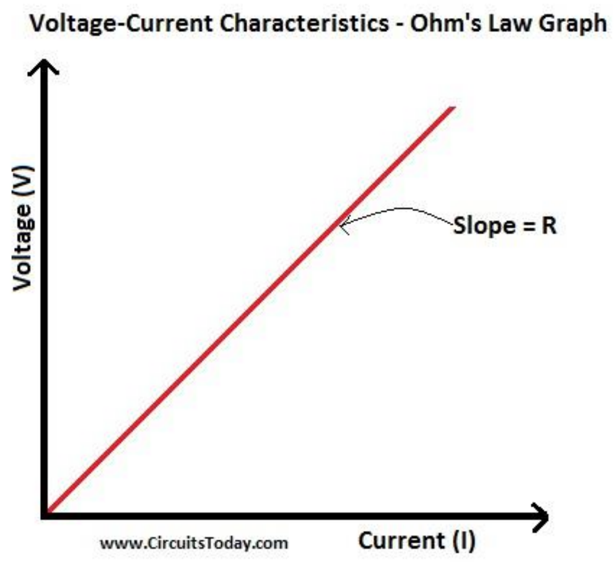
- Linearity and Superposition Principle

➢ **Linear** means something related to a line

or

➢ Linearity: is the property of a mathematical relationship (function) that can be graphically represented as a straight line





❖ The systems follow such straight line path are known as linear systems.

- The differential equation for SHM

$$\frac{d^2x}{dt^2} + w^2x = 0$$

Where $\frac{d^2x}{dt^2}$ is proportional to –x

- This equation does not contain any term such as x^2, x^3 $etc.$

Such equations are known as linear equation.

- Again, in the above equation no extra term is present which is independent of x, such equations are called as homogeneous equations.

$$\frac{d^2x}{dt^2} + w^2x + C = 0$$

Where C is independent of x

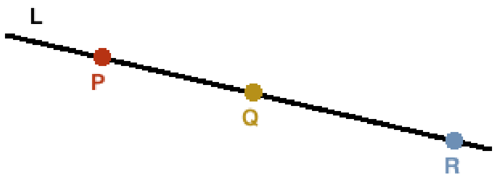
Superposition principle

- The solutions of linear systems obey the principle of superposition.

- Superposition property/principle, states that, for all linear systems, the net response caused by two or more /actions/responses/stimuli is the sum of the responses that would have been caused by each stimulus individually.

- The sum of the solutions of any linear equation is also another solution of that linear equation.

Superposition of collinear harmonic oscillator



Collinear points P, Q, and R.

- Same frequencies

Suppose we have two SHMs of equal frequencies but having different amplitudes and different phase constants acting on a system in the x-direction. The displacements x1 and x2 of the two harmonic motions, of the same angular frequency ω, differing by phase δ are given by

$$x_1 = A_1 \sin wt$$
$$x_2 = A_2 \sin(wt + \delta)$$

We are going to calculate the resultant displacement of the two H.O.

Applying superposition principle, which states that the resultant displacement equal to the vector sum (algebraic sum in this case, because the direction of the two individual oscillation is in the x-direction) of the individual displacements. Therefore, we can write

$$x = x_1 + x_2$$

$$x = A_1 sin\omega t + A_2 sin(\omega t + \delta)$$

$$= A_1 sin\omega t + A_2 sin\omega t\ cos\delta + A_2 cos\omega t\ sin\delta$$ ======

$$= (A_1 + A_2 \cos\delta)\ sin\omega t + (A_2 sin\delta) cos\omega t$$

$$= C\ sin\omega t + D \cos \omega t$$

where $C = (A_1 + A_2 \cos\delta)$ $and\ D = (A_2 sin\delta)$

$$x = \sqrt{C^2 + D^2}\left[\frac{C}{\sqrt{C^2 + D^2}} sin\omega t + \frac{D}{\sqrt{C^2 + D^2}} cos\omega t\right]$$

Put, $sin\alpha = \frac{D}{\sqrt{C^2+D^2}}$ and $cos\alpha = \frac{C}{\sqrt{C^2+D^2}}$

Therefore, we get $x = \sqrt{C^2 + D^2}[cos\alpha\ sin\omega t + sin\alpha\ cos\omega t]$

$$x = Asin(\omega t + \alpha)$$

$$A = \sqrt{C^2 + D^2} = \sqrt{(A_1 + A_2 cos\delta)^2 + (A_2 sin\delta)^2} = \sqrt{A_1^2 + A_2^2 + A_1 A_2 cos\delta}$$

$$tan\alpha = \frac{D}{C} = \frac{A_2 sin\delta}{A_1 + A_2 cos\delta}$$

Special cases

- If $\delta=\pm 2n\pi$, n=0,1,2,3,.....

$A=A_1+A_2$

- If $\delta=\pm(2n+1)\pi$, n=0,1,2,3,.....

$A= A_1-A_2$

- Different frequencies

Suppose we have two collinear harmonic oscillations of different frequencies and different amplitudes and the same phase constant (= zero).

$$x_1 = A_1 sin\,\omega_1 t$$
$$x_2 = A_2 sin\,\omega_2 t$$

The resultant of these two oscillation is given by

$$x = A_1 sin\,\omega_1 t + A_2 sin\,\omega_2 t$$

Define $\omega_1 = \omega - \Delta\omega$ and $\omega_1 = \omega + \Delta\omega$

Therefore $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ Av. frequency and
$\Delta\omega = \frac{1}{2}(\omega_1 - \omega_2)$ modulation frequency

Substituting,

$$x = A_1 \sin\omega_1 t + A_2 \sin\omega_2 t$$

$$= A_1 sin\,(\omega - \Delta\omega)t + A_2 sin\,(\omega + \Delta\omega)t$$

$$= A_1 sin\,(\omega - \Delta\omega)t + A_2 sin\,(\omega + \Delta\omega)t$$

$$= A_1(\sin\omega t\; cos\Delta\omega t - cos\omega t\; sin\Delta\omega t) + A_2(\sin\omega t\; cos\Delta\omega t + cos\omega t\; sin\Delta\omega t)$$

$$= (A_1 + A_2)\sin\omega t\; cos\Delta\omega t - (A_1 - A_2) cos\omega t\; sin\Delta\omega t$$

Define

$$(A_1 + A_2) cos\Delta\omega t = A cos\alpha$$

$$-(A_1 - A_2) sin\Delta\omega t = A\; sin\alpha$$

$$x = A cos\alpha\; sin\omega t + A sin\alpha\; cos\omega t = A sin(\omega t + \alpha)$$

A= Resultant Amplitude, α= Resultant Phase const. This oscillation can, at best, be described as periodic with an angular frequency of $\omega = \frac{1}{2}(\omega_1 + \omega_2)$, the average of the two componentfrequencies.

$$A = \sqrt{(A_1 + A_2)^2 cos^2\Delta\omega t + (A_1 - A_2)^2 sin^2\Delta\omega t}$$

$$= \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2\cos(2\Delta\omega t)}$$

And $\quad tan\alpha = \frac{-(A_1 - A_2)sin\Delta\omega t}{(A_1 + A_2)cos\Delta\omega t}$

Both A and α vary with time

If $\omega_1 \approx \omega_2$, $\Delta\omega \rightarrow 0$, modulated frequency $\Delta\omega \ll \omega$, average frequency. Modulated amplitude (resultant amplitude) A and the modulated phase constant (resultant phase constant) vary only slightly with time and may be treated as almost constant. Then

$$x = Asin(\omega t + \alpha)$$

represent an approximate harmonic oscillations having angular frequency ω. The resulting oscillations, in the case when the two frequencies of the SHMs are nearly equal, exhibit what are called beats.

Superposition of n collinear harmonic oscillator

Let us consider n collinear harmonic oscillators with same frequency, amplitude and same phase difference having displacements x1, x2, x3……..xn etc.

$$x_1 = a\sin wt$$

$$x_2 = a\sin(wt + \phi)$$

$$x_3 = a\sin(wt + 2\phi)$$

$$x_n = a\sin(wt + (n-1)\phi)$$

Applying superposition principle, the total displacement for all the H.O.

$$X = x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + x_n$$

$$X = a\sin wt + a\sin(wt + \phi) + a\sin(wt + 2\phi) \ldots\ldots\ldots\ldots\ldots + a\sin(wt + (n-1)\phi)$$

$$= a\sin wt + a\sin wt\cos\phi + a\cos wt\sin\phi + a\sin wt\cos 2\phi + a\cos wt\sin 2\phi + \ldots\ldots + a\sin wt\cos(n-1)\phi + a\cos wt\sin(n-1)\phi$$

$$= a\sin wt(1 + \cos\phi + \cos 2\phi + \ldots\ldots\ldots + \cos(n-1)\phi) + a\cos wt(\sin\varphi + \sin 2\varphi + \ldots\ldots + \sin(n-1)\varphi)$$

We define

$$a(1+\cos\phi+\cos 2\phi+\ldots\ldots\ldots\ldots+\cos(n-1)\phi) = A cos\theta$$

$$a\ (\sin\varphi+\sin 2\varphi+\ldots\ldots+\sin(n-1)\varphi) = A sin\theta$$

$$X = A\cos\theta\sin wt + A\sin\theta\cos wt$$

$$X = A\sin(wt+\phi)$$