Weak Law of Large Number (WLLN):-

Statement:- Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables and $\mu_1, \mu_2, \ldots, \mu_n$ be their respective expectations or means and let

$$B_n = Var(X_1 + X_2 + \ldots + X_n) < \infty.$$

Then,

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \frac{\mu_1 + \mu_2 + \ldots + \mu_n}{n}\right| < \varepsilon\right\} \geq 1 - \eta \quad \longrightarrow \text{(I)}$$

for all $n > n_0$, ε & η are arbitary small positive numbers, provided $\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$.

Pf:- Using Chebychev's inequality to the r.v. $\left(\frac{X_1+X_2+\dots+X_n}{n}\right)$, we get for $\varepsilon>0$,

$$P\left\{\left|\frac{X_1+X_2+\dots+X_n}{n} - E\left(\frac{X_1+X_2+\dots+X_n}{n}\right)\right|<\varepsilon\right\} \geq 1-\frac{B_n}{n^2\varepsilon^2}$$

(*) Chebychev's inequality:-

$P\{|X-M|<K\sigma\} \geq 1-\frac{1}{K^2}$. Here $K\sigma=\varepsilon \Rightarrow K=\frac{\varepsilon}{\sigma}$
$\Rightarrow \sigma=\varepsilon/K$

Here $\sigma^2=\text{Var}(X_1+\dots+X_n)=B_n$

$1-\frac{1}{K^2}=1-\frac{1}{(\varepsilon/\sigma)^2}=1-\frac{\sigma^2}{\varepsilon^2}=1-\frac{B_n}{n^2\varepsilon^2}$

[as we compute it for $\frac{X_1+X_2+\dots+X_n}{n}$]

$$\Rightarrow P\left\{\left|\frac{X_1+X_2+\dots+X_n}{n}-\frac{\mu_1+\mu_2+\dots+\mu_n}{n}\right|<\varepsilon\right\} \geq 1-\frac{B_n}{n^2\varepsilon^2}$$

$\because \varepsilon$ is arbitary, we assume $\frac{B_n}{n^2\varepsilon^2}\to 0$ as $n\to\infty$. Thus having chosen two arbitary, small positive numbers ε & η, number n_0 can be found so that $\frac{B_n}{n^2\varepsilon^2}<\eta$ will hold for $n>n_0$. Consequently, we have

$$P\left\{\left|\frac{X_1+X_2+\dots+X_n}{n}-\frac{\mu_1+\mu_2+\dots+\mu_n}{n}\right|<\varepsilon\right\} \geq 1-\eta$$

for $n>n_0$.

This statement is known as WLLN//

Thus WLLN gives $\frac{B_n}{n^2}\to 0$ as $n\to\infty$.

Remarks:- (1) WLLN can also be stated as:-

$\overline{X}_n \xrightarrow{p} \overline{\mu}_n$ provided $\frac{B_n}{n^2}\to 0$ as $n\to\infty$.

(2) For existence of WLLN, the following conditions must be satisfied:- (a) $E(X_i)$ exists for all i.

(b) $\sum V(X_i)=B_n$ exists (c) $\frac{B_n}{n^2}\to 0$ as $n\to\infty$.

condition (a) is necessary, whereas (c) is sufficient

WLLN for independent and identical r.v's :- Let $X_1, X_2, \dots X_n$ are independent and identical (i.i.d.) r.v's with mean $E(X_i) = \mu$ (for all $i = 1, 2, \dots, n$) and $var(X_i) = \sigma^2$ (for all $i = 1, 2, \dots, n$)

then $B_n = Var(X_1 + X_2 + \dots + X_n) = \sum_{i=1}^{n} V(X_i)$

$$= \sum_{i=1}^{n} \sigma^2 = n\sigma^2.$$

Then WLLN for these X_i's is given by,

$$P\left\{\left|\frac{X_1+X_2+\dots+X_n}{n} - E\left(\frac{X_1+X_2+\dots+X_n}{n}\right)\right| < \varepsilon\right\} > 1-\eta,$$

for $n > n_0$

$$\Rightarrow P\left\{\left|\frac{X_1+\dots+X_n}{n} - \frac{1}{\cancel{n}}\cdot \cancel{n}\mu\right| < \varepsilon\right\} > 1-\eta$$

or $P\left\{\left|\frac{X_1+\dots+X_n}{n} - \mu\right| < \varepsilon\right\} > 1-\eta$

In this case $\frac{B_n}{n^2} = \frac{\cancel{n}\sigma^2}{\cancel{n}n} = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$.

i.e. $P\left\{\left|\frac{X_1+\dots+X_n}{n} - \mu\right| < \varepsilon\right\} \to 1$ as $n \to \infty$

or $P\left\{\left|\frac{X_1+\dots+X_n}{n} - \mu\right| \geq \varepsilon\right\} \to 0$ as $n \to \infty$.

<u>Theorem</u>:- If the variables are uniformly bounded then the condition $\lim_{n\to\infty} \frac{B_n}{n^2} = 0$ is necessary and sufficient for WLLN to hold.

<u>Pf</u>:- Let $\xi_i = X_i - a_i$ where $E(X_i) = a_i$, then

$$E(\xi_i) = E(X_i) - a_i = a_i - a_i = 0. \quad (i = 1, 2, \dots, n)$$

$\because X_i$'s are uniformly bounded, there exists a positive number $c < \infty$ such that $|\xi_i| < c$.

If $p = P\{|\xi_1 + \xi_2 + \dots + \xi_n| \le n\epsilon\}$, then

$1 - p = P\{|\xi_1 + \xi_2 + \dots + \xi_n| > n\epsilon\}$.

Let $U_n = \xi_1 + \xi_2 + \dots + \xi_n$, then $E[U_n] = \sum_{i=1}^{n} E(\xi_i) = 0$

and $var(U_n) = B_n$ (say)

Also $|U_n| = \left|\sum_{i=1}^{n} \xi_i\right| \le \sum_{i=1}^{n} |\xi_i| < \sum_{i=1}^{n} c = nc$

and $P(|U_n| \le n\epsilon) = p$ & $P\{|U_n| > n\epsilon\} = 1-p$.

$\Rightarrow B_n = \int_0^\infty u_n^2 \, dF(u_n)$ where $f(u_n)$ is the density function of u_n

$= \int_{u_n^2 \le n^2\epsilon^2} U_n^2 \, dF + \int_{u_n^2 > n^2\epsilon^2} U_n^2 \, dF$ ($F(u_n)$ is the distribution function)

$\Rightarrow B_n \le n^2\epsilon^2 \int dF + n^2 c^2 \int dF$

$\le n^2 \epsilon^2 p + n^2 c^2 (1-p)$

$\therefore B_n/n^2 \le \epsilon^2 p + c^2(1-p)$

If the WLLN holds, $1 - p = P\{|\xi_1 + \xi_2 + \dots + \xi_n| > n\epsilon\} \to 0$ as $n \to \infty$

Hence as $n \to \infty$, $(1-p) \to 0$ and $\frac{B_n}{n^2} < \epsilon^2 p + c^2 \delta$, where ϵ & δ being arbitarily small positive numbers.

Hence $\frac{B_n}{n^2} \to 0$ as $n \to \infty$ //