Ex) Solve the system of eqⁿs

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20$$

by factorization method.

Solⁿ:- The given system of eqⁿs can be written in the form $AX = B$ ---(1)

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$

Now, let $A = LU$, where L is the lower triangular matrix and U is the upper triangular matrix. Thus we have,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating the corresponding elements, we get,

$u_{11} = 1$; $u_{12} = 2$; $u_{13} = 3$;

$l_{21} u_{11} = 2$
$\Rightarrow l_{21} = 2$; $l_{21} u_{12} + u_{22} = 5$
$\Rightarrow 2.2 + u_{22} = 5$
$\Rightarrow u_{22} = 1$;

$l_{21} u_{13} + u_{23} = 2$
$\Rightarrow 2.3 + u_{23} = 2$
$\Rightarrow u_{23} = -4$; $l_{31} u_{11} = 3$
$\Rightarrow l_{31} . 1 = 3$
$\Rightarrow l_{31} = 3$;

$l_{31} u_{12} + l_{32} u_{22} = 1$
$\Rightarrow 3.2 + l_{32} . 1 = 1$
$\Rightarrow l_{32} = -5$

and $l_{31} u_{13} + l_{32} u_{23} + u_{33} = 5$
$\Rightarrow 3.3 + (-5)(-4) + u_{33} = 5$
$\Rightarrow u_{33} = -24$

Hence the given system of $AX = B$ is equivalent to $LUX = B$ i.e

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

Now, let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix},$$

giving $y_1 = 14$; $2y_1 + y_2 = 18$; $3y_1 - 5y_2 + y_3 = 20$
$\Rightarrow y_2 = 18 - 2 \times 14$ $\Rightarrow y_3 = 5y_2 + 20 - 3y_1$
$= -10$ $= -72$

Now,

$$UX = Y$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

giving

$$-24x_3 = -72$$

$$\Rightarrow x_3 = 3$$

$$x_2 - 4x_3 = -10$$

$$\Rightarrow x_2 = -10 + 4x_3 = -10 + 12$$

$$= 2$$

and

$$x_1 + 2x_2 + 3x_3 = 14$$

$$\Rightarrow x_1 = 14 - 3x_3 - 2x_2$$

$$= 14 - 9 - 4$$

$$= 1$$

Hence the sol^n of the given eq^{n}s is

$$x_1 = 1, \; x_2 = 2, \; x_3 = 3$$