

Corollary: If a and b are integers, with $b \neq 0$, then there exist unique integers q and r such that

$$a = qb + r, \quad 0 \leq r < |b|$$

Pf. Case 1. If $b > 0$; then from division algorithm \exists unique integers q and r such that

$$a = qb + r, \quad 0 \leq r < b = |b| \quad [\because b > 0 \quad \therefore |b| = b]$$

Case 2. If $b < 0$. Then $|b| > 0$. Then by division algorithm \exists unique integers q' and r for which

$$a = q'(|b|) + r, \quad 0 \leq r < |b| \quad \text{--- (1)}$$

But since $b < 0$ so $|b| = -b$.

$$\therefore (1) \Rightarrow a = q'(-b) + r, \quad 0 \leq r < |b|$$

$$= (-q')b + r, \quad 0 \leq r < |b|$$

$$= qb + r, \quad 0 \leq r < |b|, \quad q = -q'$$

(H.W) Ex. Prove that if a and b are integers with $b > 0$, then there exists integers q and r satisfying

$$a = qb + r, \text{ where } 2b \leq r < 3b$$

Hint: Consider the set

$$S = \{a - xb \mid x \text{ is an integer}, a - xb \geq 2b\}$$

Continued as the proof of Division Algorithm.

Ex. Show that square of any integer is either of the form $4k$ or $4k+1$.

Soln: Let a be any integer. Then for $b = 2$, from division algorithm it is clear that when a is divided

$$a = q \cdot 2 + r, \quad 0 \leq r < 2 \quad \text{i.e. } r = 0, 1$$

$$\Rightarrow a = 2q + r, \quad r = 0, 1$$

$$\text{for } r = 0, \quad a = 2q + 0 = 2q \Rightarrow a^2 = 4q^2 = 4k, \quad k = q^2$$

$$\text{for } r = 1, \quad a = 2q + 1 \Rightarrow a^2 = 4q^2 + 4q + 1 = 4(q^2 + q) + 1 \\ = 4k + 1, \quad k = q^2 + q$$

Ex: Prove that square of any odd integer
is of the form $8k+1$.

Sol: Let a be any odd integer. For $b=4$,
from division algorithm of integers q and n we get

$$a = q \cdot 4 + r, \quad 0 \leq r < 4$$

$$\Rightarrow a = 4q + r, \quad r = 0, 1, 2, 3.$$

i.e. any integer a is of the form $4q+0$,
 $4q+1$, $4q+2$, $4q+3$ i.e. $4q$, $4q+1$, $4q+2$,
 $4q+3$. Only those integers of the form
 $4q+1$ and $4q+3$ are odd.

$$\text{for } a = 4q+1 \Rightarrow a^2 = (4q+1)^2 = 16q^2 + 8q + 1$$

$$= 8(2q^2 + q) + 1$$

$$= 8k + 1$$

$$\text{for } a = 4q+3 \Rightarrow a^2 = (4q+3)^2$$

$$= 16q^2 + 24q + 9$$

$$\Rightarrow a^2 = 16q^2 + 24q + 8 + 1$$

$$= 8(2q^2 + 3q + 1) + 1$$

$$= 8k + 1$$

Ex: Show that any integer of the form $6k+5$
is also of the form $3j+2$ but not conversely.

Ex: Use the Division Algorithm to establish the following:

(a) The square of any integer is either of the form
 $3k$ or $3k+1$.

(b) The cube of any integer is either of the forms
 $9k$, $9k+1$ or $9k+8$.

(c) The fourth power of any integer is either of
the form $5k$ or $5k+1$.

Ex: prove that no integer in the following sequence
is a perfect square:

11, 111, 1111, 11111, ...

Sol: A typical term 111...111 can be written as

$$111\ldots111 = 111\ldots108 + 3 = 4k + 3.$$

But we know square of any integer is either of
the form $4k$ or $4k+1$. So no integer in the
given sequence is a perfect square.

Ex: prove that the expression $a(a^n+2)/3$ is an
integer for all integers $a > 1$.

Sol: Let $a > 1$ be any integer. For $b = 3$,

3 unique integers q and r S.t.

$$a = q \cdot 3 + r, \quad 0 \leq r < 3 \text{ i.e. } r=0,1,2$$

$$\text{for } r=0, \quad a = 3q + 0 = 3q$$

$$\text{Then } \frac{a(a^n+2)}{3} = \frac{3q(9q^n+2)}{3} = q(9q^n+2), \text{ an integer}$$

since q is an integer.

$$\text{for } r=1, \quad a = 3q + 1$$

$$\text{Then } \frac{a(a^n+2)}{3} = \frac{(3q+1)(9q^n+6q+1+2)}{3} = \frac{(3q+1) \cdot 3(3q^n+2q+1)}{3}$$

$$= (3q+1)(3q^n+2q+1), \text{ an integer.}$$

$$\text{for } r=2, \quad a = 3q + 2$$

Then

$$\begin{aligned} \frac{a(a^n+2)}{3} &= \frac{(3q+2)(9q^n+12q+4+2)}{3} = \frac{(3q+2)(9q^n+12q+6)}{3} \\ &= \frac{(3q+2) \cdot 3(3q^n+4q+2)}{3} \\ &= (3q+2)(3q^n+4q+2), \text{ an integer.} \end{aligned}$$

Ex: For $n > 1$, prove that $n(n+1)(2n+1)/6$ is an
integer.