Peano's Axiom or Peano's postulate

$\mathbb{N} \to$ set of natural numbers.

(P1). $1 \in \mathbb{N}$ i.e. N is a non-empty set and contains an element we designate as 1.

(P2) For each element $n \in \mathbb{N}$ there is a unique element $n_0 \in \mathbb{N}$ called the successor of n.

(P3) For each $n \in \mathbb{N}$, $n_0 \neq 1$ i.e. 1 is not the successor of any element in $\mathbb{N}$.

(P4) for each pair $n, m \in \mathbb{N}$ with $n \neq m$, $n_0 \neq m_0$ i.e. distinct elements in $\mathbb{N}$ have distinct successors.

(P5) If $A \subseteq \mathbb{N}$, $1 \in A$ and $p \in A$ implies $p_0 \in A$ then $A = \mathbb{N}$, where p_0 is the successor of p.

Note. (P5) is called the principle of Mathematical induction.

Well ordering principle. Every nonempty set S of nonnegative integers contains a least element, that is there is some integer a in S such that $a \leq b$ for all $b \in S$.

Theorem: Archimedean property. If a and b are any positive integers, then there exists a positive integer n such that $na \geq b$.

Pf. Assume that the statement of the theorem is not true, so that for some a and b, $na < b$ for every positive integer n.

Consider the set $S = \{ b - na \mid n \text{ a positive integer} \}$

As $na < b$, $\forall$ positive integer n so $b - na > 0$.

Hence S consists entirely positive integers.

So by well ordering principle S contains a least element say $b - ma$ where m is ~~some~~ a positive integer.

Now m is positive integer $\Rightarrow m+1$ is also positive integer.

$\therefore b - (m+1)a \in S$ since S contains elements of the form $b - na$.

Now,

$$b - (m+1)a = (b - ma) - a < b - ma$$

$$\Rightarrow b - (m+1)a < b - ma$$

which is a contradiction because $b - ma$ is the least element of S. Hence our assumption ~~is wrong~~ that Archimedean property did not hold is wrong. ~~Hence proved~~ ~~the~~ Hence the Archimedean property is proved.